

LAMBDA DETERMINANTS

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ABSTRACT.

In this paper we prove a homogenous generalization of the lambda determinant formula of Mills, Robbins and Rumsey. In our formula the parameters depends on two indices. Our result also extends a recent formula of Di Francesco.

1. INTRODUCTION

An *alternating sign matrix* is a square matrix of 0's 1's and -1 's such that the sum of each row and column is 1 and the non-zero entries in each row and column alternate in sign. For example:

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Alternating sign matrices arise naturally in Dodgson's condensation method for calculating λ -determinants [Bre99].

For each $k = 0..n$ let us denote by $x[k]$ the doubly indexed collection of variables $x[k]_{i,j}$ with indices running from $i, j = 1..(n - k + 1)$. One should think of these variables as forming a square pyramid with base of dimension $n + 1$ by $n + 1$. The index k determines the "height" of the variable in the pyramid.

The variables $x[0]$ and $x[1]$ are to be thought of as initial conditions. The remaining $x[k]$ are defined in terms of the following *octahedral recurrence*:

$$(1) \quad x[k+1]_{i,j} = \frac{\mu_{i,n-k-j}x[k]_{i,j}x[k]_{i+1,j+1} + \lambda_{i,j}x[k]_{i,j+1}x[k]_{i+1,j}}{x[k-1]_{i+1,j+1}}$$

The main result of this paper is a closed form expression for $x[k]_{1,1}$. Our result generalizes the result obtained by Di Francesco [Fra12], who considered coefficients $\lambda_{ij} \equiv \lambda_{i-j}$ and $\mu_{ij} \equiv \mu_{i-j}$.

The outline of this paper is as follows. We begin with some definitions which are necessary in order to write down the closed form expression. In section 3 we introduce *left cumulant matrices* and a

pair of *up / down operators*. In section 4 we introduce *right cumulant matrices* and a second pair of up / down operators which are closely related to the first. Finally in section 5 we prove our main theorem.

2. CLOSED FORM EXPRESSION

For each n by n alternating sign matrix B let \overline{B} be the matrix whose (i, j) -th entry is equal to the sum of the entries lying above and to the left of the (i, j) -th entry of B . Similarly, let \underline{B} be the matrix whose (i, j) -th entry is equal to the sum of the entries lying above and to the right of the (i, j) -th entry of B . For example:

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \overline{X} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \underline{X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

We shall refer to \overline{B} as the *left cumulant* matrix of B and \underline{B} as the *right cumulant* matrix of B . The original alternating sign matrix may be recovered by the formula:

$$(2) \quad B_{ij} = \overline{B}_{ij} + \overline{B}_{i-1,j-1} - \overline{B}_{i,j-1} - \overline{B}_{i-1,j}$$

$$(3) \quad = \underline{B}_{ij} + \underline{B}_{i-1,j+1} - \underline{B}_{i,j+1} - \underline{B}_{i-1,j}$$

If the indices are out of range, then the value of B_{ij} is taken to be zero.

Lemma 2.1. *If B' is the alternating sign matrix obtained from B by multiplying on the right by the maximum permutation then \underline{B}' is the matrix obtained from \overline{B} by multiplying on the right by the maximum permutation.*

We shall make use of the notation:

$$\underline{B} = (\overline{B'})'$$

Lemma 2.2. *For all i, j we have:*

$$\overline{B}_{i,j} + \underline{B}_{i,j+1} = i$$

Proof. The left hand side is equal to the sum of all the entries of the alternating sign matrix B in the first i rows. Since the sum of entries in each row of B is equal to 1, the final result is equal to i as claimed. \square

Comparing matrices entrywise, the \overline{B} of size n form a lattice. We remark that this lattice coincides with the completion of the Bruhat order to alternating sign matrices as carried out in Lascoux and Schützenberger [LS96]. One can apply the same operation with \underline{B} to form a dual lattice.

Let us define the *lambda weight* of a k by k alternating sign matrix B to be:

$$\lambda^{F(B)} = \lambda^{\overline{I}-\overline{B}} = \prod_{i,j=1}^k \lambda_{i,j}^{\min(i,j)-\overline{B}_{i,j}}$$

Similarly, let us define the *mu weight* of an k by k alternating sign matrix B to be:

$$\mu^{G(B)} = \mu^{\overline{I}-\overline{B}} = \prod_{i,j=1}^k \mu_{i,j}^{\max(i-j+1,0)-\overline{B}_{i,k+1-j}}$$

The *standard weight* of an alternating sign matrix B is simply:

$$x^B = \prod_{i,j=1}^k x_{i,j}^{B_{i,j}}$$

Robbins and Rumsey [RR86] defined two multiplicity free operators acting on the vector space spanned by alternating sign matrices, which we shall discuss in section 3:

$$\mathfrak{U} : \text{ASM}(n) \rightarrow \mathbb{Z}[\text{ASM}(n+1)]$$

$$\mathfrak{D} : \text{ASM}(n) \rightarrow \mathbb{Z}[\text{ASM}(n-1)]$$

Our closed form expression for $x[k]_{1,1}$ now takes the form;

$$(4) \quad \boxed{x[k]_{1,1} = \sum_{\substack{(A,B) \\ |B|=k, |A|=k-1 \\ A \in \mathfrak{D}(B)}} \lambda^{F(B)} s(\lambda)^{-F(A)} \mu^{G(B)} t(\mu)^{-G(A)} x[1]^B s(x[0])^{-A}}$$

where:

$$(5) \quad s(z)_{i,j} = z_{i+1,j+1}$$

$$(6) \quad t(z)_{i,j} = z_{i+1,j-1}$$

Note that this formula shows that $x[k]_{1,1}$ is a Laurent polynomial, and not just a rational function as would be expected from its recursive definition. This is an example of the Laurent phenomenon. See, for example [FZ02].

3. UP AND DOWN OPERATORS

We shall now define the multiplicity free operators acting on the vector space spanned by alternating sign matrices mentioned in the previous section:

$$\begin{aligned}\mathfrak{U} : \text{ASM}(n) &\rightarrow \mathbb{Z}[\text{ASM}(n+1)] \\ \mathfrak{D} : \text{ASM}(n) &\rightarrow \mathbb{Z}[\text{ASM}(n-1)]\end{aligned}$$

These operators have the property that $B \in \text{ASM}(n)$ contains r ones and s negative ones then number of terms occuring in $\mathfrak{U}(B)$ is 2^r while the number of terms occuring in $\mathfrak{D}(B)$ is 2^s .

If we fix an order on the -1 's of B then each element A of $\mathfrak{D}(B)$ is naturally indexed by a binary string. Similarly if we fix an order of the 1 's in B then each element C of $\mathfrak{U}(B)$ is indexed by a binary string.

To define these operators we shall need the notion of *left interlacing matrices*:

$$\begin{pmatrix} \overline{B}_{1,1} & \overline{A}_{1,1} & \overline{B}_{1,2} & \overline{A}_{1,2} & \overline{B}_{1,3} & \overline{A}_{1,3} & \overline{B}_{1,4} \\ \overline{B}_{2,1} & \overline{A}_{2,1} & \overline{B}_{2,2} & \overline{A}_{2,2} & \overline{B}_{2,3} & \overline{A}_{2,3} & \overline{B}_{2,4} \\ \overline{B}_{3,1} & \overline{A}_{3,1} & \overline{B}_{3,2} & \overline{A}_{3,2} & \overline{B}_{3,3} & \overline{A}_{3,3} & \overline{B}_{3,4} \\ \overline{B}_{4,1} & \overline{A}_{4,1} & \overline{B}_{4,2} & \overline{A}_{4,2} & \overline{B}_{4,3} & \overline{A}_{4,3} & \overline{B}_{4,4} \end{pmatrix}$$

The conditions on the matrix \overline{A} are as follows:

$$\begin{pmatrix} x & y \\ & a \\ z & w \end{pmatrix} \quad \boxed{x, w-1 \leq a \leq y, z}$$

An example:

$$\begin{pmatrix} 0 & & 1 & & 1 & & 1 \\ & \{0, 1\} & & 1 & & 1 & \\ 1 & & 1 & & 2 & & 2 \\ & 1 & & \{1, 2\} & & 2 & \\ 1 & & 2 & & 2 & & 3 \\ & 1 & & 2 & & 3 & \\ 1 & & 2 & & 3 & & 4 \end{pmatrix}$$

Above and to the left of a -1 in the alternating sign matrix B there are two possible choices for the corresponding value of the left cumulant matrix \overline{A} . At all other positions there is a single choice [RR86].

$$\begin{array}{ccc}
& \overline{A}_{11} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} & \\
\swarrow & & \searrow \\
\overline{A}_{01} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} & & \overline{A}_{10} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \\
\swarrow & & \searrow \\
& \overline{A}_{00} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} &
\end{array}$$

Here are the corresponding alternating sign matrices:

$$\begin{array}{ccc}
& A_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \\
\swarrow & & \searrow \\
A_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & & A_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\swarrow & & \searrow \\
& A_{00} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} &
\end{array}$$

One may check that adding one at position (i, j) in the left cumulant matrix \overline{A} is equivalent to adding the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ with upper left hand corner at position (i, j) to the corresponding alternating sign matrix A [RR86].

In our example we have:

$$\mathfrak{D}(X) = A_{0,0} + A_{1,0} + A_{0,1} + A_{1,1}$$

We shall be especially interested in the “smallest” matrix A which is left interlacing with the matrix B and which we denote by $A^{\min} = A_{00\dots 0}$. We have, by construction:

$$\boxed{\overline{A}_{ij}^{\min} = \max(\overline{B}_{ij}, \overline{B}_{i+1,j+1} - 1)}$$

The \mathfrak{U} operator is defined similarly.

$$\begin{pmatrix} \overline{C}_{1,1} & \overline{B}_{1,1} & \overline{C}_{1,2} & \overline{B}_{1,2} & \overline{C}_{1,3} \\ \overline{C}_{2,1} & \overline{B}_{2,1} & \overline{C}_{2,2} & \overline{B}_{2,2} & \overline{C}_{2,3} \\ \overline{C}_{3,1} & & \overline{C}_{3,2} & & \overline{C}_{3,3} \end{pmatrix}$$

The rule for constructing all possible matrices \overline{C} for a given matrix \overline{B} is the last row and last column must be strictly increasing from 1 to n as well as that:

$$\begin{pmatrix} x & y \\ & \textcolor{blue}{c} \\ z & w \end{pmatrix} \quad \boxed{y, z \leq \textcolor{blue}{c} \leq w, x + 1}$$

Here is an example:

$$Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \overline{Y} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

Interlacing matrices:

$$\begin{pmatrix} 0 & & \{0, 1\} & 1 \\ & 0 & & 1 \\ \{0, 1\} & & 1 & 2 \\ & 1 & & 2 \\ 1 & & 2 & 3 \end{pmatrix}$$

Above and to the left of a 1 in the alternating sign matrix B there are two possible choices for the corresponding value of \overline{C} . At all other positions there is a single choice [RR86].

$$\begin{array}{ccc} & \overline{C}_{11} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} & \\ \swarrow & & \searrow \\ \overline{C}_{01} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} & & \overline{C}_{10} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \\ \searrow & & \swarrow \\ & \overline{C}_{00} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} & \end{array}$$

Here are the corresponding alternating sign matrices:

$$\begin{array}{ccc}
 & C_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \\
 \swarrow & & \searrow \\
 C_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} & & C_{10} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
 \swarrow & & \searrow \\
 & C_{00} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} &
 \end{array}$$

One may check that, as expected, subtracting one at position (i, j) in the left cumulant matrix \overline{C} is equivalent to subtracting the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ with upper left hand corner at position (i, j) from the corresponding alternating sign matrix C [RR86].

In our example we have:

$$\mathfrak{U}(Y) = C_{0,0} + C_{0,1} + C_{1,0} + C_{1,1}$$

This time we shall be especially interested in the “largest” matrix C which is left interlacing with B and which we denote by $C^{\max} = C_{11\dots 1}$. We have, by construction:

$$\boxed{\overline{C}_{ij}^{\max} = \min(\overline{B}_{ij}, \overline{B}_{i-1,j-1} + 1)}$$

4. MORE UP-DOWN OPERATORS

We will need a second set of up and down operators which are closely related to the first.

$$\mathfrak{U}^* : \text{ASM}(n) \rightarrow \mathbb{Z}[\text{ASM}(n+1)]$$

$$\mathfrak{D}^* : \text{ASM}(n) \rightarrow \mathbb{Z}[\text{ASM}(n-1)]$$

To define these operators we make use of *right interlacing matrices*:

$$\begin{pmatrix} \underline{B}_{1,1} & \underline{A}_{1,1}^* & \underline{B}_{1,2} & \underline{A}_{1,2}^* & \underline{B}_{1,3} & \underline{A}_{1,3}^* & \underline{B}_{1,4} \\ \underline{B}_{2,1} & \underline{A}_{2,1}^* & \underline{B}_{2,2} & \underline{A}_{2,2}^* & \underline{B}_{2,3} & \underline{A}_{2,3}^* & \underline{B}_{2,4} \\ \underline{B}_{3,1} & \underline{A}_{3,1}^* & \underline{B}_{3,2} & \underline{A}_{3,2}^* & \underline{B}_{3,3} & \underline{A}_{3,3}^* & \underline{B}_{3,4} \\ \underline{B}_{4,1} & \underline{A}_{4,1}^* & \underline{B}_{4,2} & \underline{A}_{4,2}^* & \underline{B}_{4,3} & \underline{A}_{4,3}^* & \underline{B}_{4,4} \end{pmatrix}$$

In the right interlacing case, the conditions on the matrix \underline{A}^* are:

$$\begin{pmatrix} x & y \\ & \textcolor{red}{a} \\ z & w \end{pmatrix} \quad \boxed{y, z - 1 \leq \textcolor{red}{a} \leq x, w}$$

Continuing with our example matrix X :

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ & \textcolor{red}{1} & \{0, 1\} & \textcolor{red}{0} & \\ 1 & 1 & 2 & & 2 \\ & \textcolor{red}{2} & 1 & \{0, 1\} & \\ 2 & 1 & 1 & & 0 \\ & \textcolor{red}{3} & 2 & 1 & \\ 4 & 3 & 2 & & 1 \end{pmatrix}$$

Above and to the *right* of a -1 in the alternating sign matrix B there are two possible choices for the corresponding value of the right cumulant matrix \underline{A}^* . Again, if we fix an order on the -1 's of B then each element A^* of $\mathfrak{D}(B)$ is naturally indexed by a binary string determining the position in the right cumulant matrix \underline{A}^* where the larger of the two possible values was chosen.

$$\begin{array}{ccc} & \underline{A}_{11}^* = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} & \\ \swarrow & & \searrow \\ \underline{A}_{01}^* = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} & & \underline{A}_{10}^* = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \\ \swarrow & & \searrow \\ & \underline{A}_{00}^* = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} & \end{array}$$

Here are the corresponding alternating sign matrices:

$$\begin{array}{ccc}
 & A_{11}^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \\
 \swarrow & & \searrow \\
 A_{01}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & & A_{10}^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 \swarrow & & \searrow \\
 & A_{00}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &
 \end{array}$$

Adding one at position (i, j) in the right cumulant matrix \underline{A}^* is equivalent to adding the matrix $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ with upper *right* hand corner at position (i, j) to the alternating sign matrix A^* .

If B is an n by n alternating sign matrix then $\mathfrak{D}^*(B)$ is the sum of all $n - 1$ by $n - 1$ alternating sign matrices A^* such that \underline{A}^* is right interlacing with \underline{B}^* .

Now for the \mathfrak{U}^* operator.

$$\begin{pmatrix} \underline{C}_{1,1}^* & \underline{C}_{1,2}^* & \underline{C}_{1,3}^* \\ \underline{C}_{2,1}^* & \underline{C}_{2,2}^* & \underline{C}_{2,3}^* \\ \underline{C}_{3,1}^* & \underline{C}_{3,2}^* & \underline{C}_{3,3}^* \end{pmatrix}$$

The rule for constructing all possible matrices \underline{C}^* for a given matrix \underline{B} is the last column must be strictly increasing from 1 to $n + 1$, the last row must be strictly decreasing from n to 1, and:

$$\begin{pmatrix} x & y \\ \textcolor{blue}{c} & w \\ z & \end{pmatrix} \quad \boxed{w, x \leq \textcolor{blue}{c} \leq y + 1, z}$$

Here is an example:

$$Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \underline{Y} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

Interlacing matrices:

$$\begin{pmatrix} 1 & 1 & \{0,1\} \\ & 1 & 1 \\ 2 & \{1,2\} & 1 \\ & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

Above and to the *right* of a 1 in the alternating sign matrix B there are two possible choices for the corresponding value of \underline{C}^* . At all other positions there is a single choice.

Fixing an order on the 1's of B , for each element C^* of $\mathfrak{A}^*(B)$ is naturally indexed by a binary string determining the position in the right cumulant matrix \underline{C}^* where the larger of the two possible values was chosen.

$$\begin{array}{ccc} & \underline{C}_{11}^* = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} & \\ \swarrow & & \searrow \\ \underline{C}_{01}^* = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} & & \underline{C}_{10}^* = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} \\ \swarrow & & \searrow \\ & \underline{C}_{00}^* = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} & \end{array}$$

Here are the corresponding alternating sign matrices:

$$\begin{array}{ccc} & C_{11}^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \\ \swarrow & & \searrow \\ C_{01}^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & & C_{10}^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ \swarrow & & \searrow \\ & C_{00}^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \end{array}$$

Subtracting one at position (i, j) in the right cumulant matrix \underline{C}^* is equivalent to subtracting the matrix $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ with upper *right* hand corner at position (i, j) from the alternating sign matrix C^* .

Proposition 4.1.

$$A_{\min}^* = A_{\max}$$

Proof. Consider the following segments of interlacing matrices:

$$\begin{pmatrix} a & b & c & d \\ & x & y & z \\ e & f & g & h \\ & w & u & v \\ i & j & k & \ell \\ & t & s & r \\ m & n & o & p \end{pmatrix} \quad \begin{pmatrix} a^* & b^* & c^* & d^* \\ & x^* & y^* & z^* \\ e^* & f^* & g^* & h^* \\ & w^* & u^* & v^* \\ i^* & j^* & k^* & \ell^* \\ & t^* & s^* & r^* \\ m^* & n^* & o^* & p^* \end{pmatrix}$$

The elements a, b, c , etc... belong to the left cumulant matrix \overline{B} while the elements x, y, z etc... belong to the left-interlacing matrix \overline{A}_{\max} .

Similarly the elements a^*, b^*, c^* , etc... belong to the right cumulant matrix \underline{B}^* while the elements x^*, y^*, z^* etc... belong to the right-interlacing matrix \underline{A}_{\min}^* .

We wish to show that the value of the entry of A_{\max} at position u is equal to the value of A_{\min}^* at position u^* . That is, by equations (2) and (3) we want to show that:

$$x + u - w - y = u^* + z^* - y^* - v^*$$

As a consequence of lemma 2.2 there is some γ such that:

$$\begin{aligned} a + b^* &= b + c^* = c + d^* = \gamma \\ e + f^* &= f + g^* = g + h^* = \gamma + 1 \\ i + j^* &= j + k^* = k + \ell^* = \gamma + 2 \\ m + n^* &= n + o^* = o + p^* = \gamma + 3 \end{aligned}$$

Now, by construction, we have:

$$\begin{aligned}
x + u - w - y &= \min(a, f - 1) + \min(f, k - 1) - \min(e, j - 1) - \min(b, g - 1) \\
&= \min(\gamma - b^*, \gamma - g^*) + \min(\gamma + 1 - g^*, \gamma + 1 - \ell^*) \\
&\quad - \min(\gamma + 1 - f^*, \gamma + 1 - k^*) - \min(\gamma - c^*, \gamma - h^*) \\
&= -\max(b^*, g^*) - \max(g^*, \ell^*) + \max(f^*, k^*) + \max(c^*, h^*) \\
&= -y^* - v^* + u^* + z^*
\end{aligned}$$

The result follows. \square

A similar argument to the above may be used to show that $A_{\max}^* = A_{\min}$ as well as $C_{\min}^* = C_{\max}$ and $C_{\max}^* = C_{\min}$. More precisely:

Proposition 4.2. *If s is a binary string, and \bar{s} is its complement, then $A_s = A_{\bar{s}}^*$ and $C_s = C_{\bar{s}}^*$.*

In other words, for any alternating sign matrix B the partial order of matrices occuring in $\mathfrak{D}^*(B)$ is precisely the dual of the partial order of matrices occuring in $\mathfrak{D}(B)$. Similarly for $\mathfrak{U}(B)$ and $\mathfrak{U}^*(B)$.

5. PROOF OF MAIN THEOREM

Our proof is almost identical to that given in [RR86]. Let us recall the recurrence:

$$(7) \quad x[k+1]_{i,j} = \frac{\mu_{i,n-k-j} x[k]_{i,j} x[k]_{i+1,j+1} + \lambda_{i,j} x[k]_{i,j+1} x[k]_{i+1,j}}{x[k-1]_{i+1,j+1}}$$

To simplify things, let us introduce the notation:

$$D(x[k])_{i,j} = \mu_{i,n-k-j} x[k]_{i,j} x[k]_{i+1,j+1} + \lambda_{i,j} x[k]_{i,j+1} x[k]_{i+1,j}$$

so that we may rewrite equation 7 as:

$$x[k+1]_{i,j} = \frac{D(x[k])_{i,j}}{s(x[k-1])_{i,j}}$$

Theorem 5.1. *For $2 \leq k \leq n$ we have:*

$$(8) \quad x[k]_{1,1} = \sum_{\substack{(A,B) \\ |B|=k, |A|=k-1}} \lambda^{F(B)} s(\lambda)^{-F(A)} \mu^{G(B)} t(\mu)^{-G(A)} x[1]^B s(x[0])^{-A}$$

The sum is over all pairs of matrix (A, B) such that A occurs in the expansion of $\mathfrak{D}(B)$.

Proof. The result is trivially true when $k = 2$. Making use of the invariance in k , followed by the recurrence, we may obtain $x[k+1]_{1,1}$ from $x[k]_{i,j}$ as follows:

$$\begin{aligned} x[k+1]_{1,1} &= \sum_{\substack{(A,B) \\ |B|=k, |A|=k-1}} \lambda^{F(B)} s(\lambda)^{-F(A)} \mu^{G(B)} t(\mu)^{-G(A)} x[2]^B s(x[1])^{-A} \\ &= \sum_{\substack{(A,B) \\ |B|=k, |A|=k-1}} \lambda^{F(B)} s(\lambda)^{-F(A)} \mu^{G(B)} t(\mu)^{-G(A)} \left(\frac{D(x[1])}{s(x[0])} \right)^B s(x[1])^{-A} \end{aligned}$$

We must show that this is equal to:

$$\sum_{\substack{(B,C) \\ |C|=k+1, |B|=k}} \lambda^{F(C)} s(\lambda)^{-F(B)} \mu^{G(C)} t(\mu)^{-G(B)} x[1]^C s(x[0])^{-B}$$

To do this, we fix some alternating sign matrix B with $|B| = k$ and take the coefficient of $s(x[0])^{-B}$ on both sides. We must now prove that:

$$\begin{aligned} (9) \quad & \sum_{|A|=k-1} \lambda^{F(B)} s(\lambda)^{-F(A)} \mu^{G(B)} t(\mu)^{-G(A)} D(x[1])^B s(x[1])^{-A} \\ &= \sum_{|C|=k+1} \lambda^{F(C)} s(\lambda)^{-F(B)} \mu^{G(C)} t(\mu)^{-G(B)} x[1]^C \end{aligned}$$

Here the sum is over all A (resp. C) which may be found in the expansion of $\mathfrak{D}(B)$ (resp $\mathfrak{U}(B)$).

Making use of proposition 4.2 we may rewrite the right hand side of equation (9) as:

$$\begin{aligned} & \sum_{|C|=k+1} \lambda^{F(C)} s(\lambda)^{-F(B)} \mu^{G(C)} t(\mu)^{-G(B)} x[1]^C \\ &= s(\lambda)^{-F(B)} t(\mu)^{-G(B)} x[1]^{C_{\max}} \lambda^{F(C_{\max})} \mu^{G(C_{\min}^*)} \prod_{B_{ij}=1} (\mu_{i,n-k-j} + \lambda_{ij} \frac{x[1]_{i+1,j} x[1]_{i,j+1}}{x[1]_{ij} x[1]_{i+1,j+1}}) \\ (10) \quad &= s(\lambda)^{-F(B)} t(\mu)^{-G(B)} x[1]^{C_{\max}} \lambda^{F(C_{\max})} \mu^{G(C_{\min}^*)} \prod_{B_{ij}=1} \frac{D(x[1]_{i,j})}{x[1]_{ij} s(x[1]_{i,j})} \end{aligned}$$

while the left hand side of equation (9) may be written as:

$$\begin{aligned}
& \sum_{|A|=k-1} \lambda^{F(B)} s(\lambda)^{-F(A)} \mu^{G(B)} t(\mu)^{-G(A)} D(x[1])^B s(x[1])^{-A} \\
&= \lambda^{F(B)} \mu^{G(B)} D(x[1])^B s(\lambda)^{-F(A_{\min})} t(\mu)^{-G(A_{\max}^*)} s(x[1])^{-A_{\min}} \\
& \quad \prod_{B_{i,j}=-1} (\mu_{i,n-k-j} + \lambda_{ij} \frac{x[1]_{i+1,j} x[1]_{i,j+1}}{x[1]_{ij} x[1]_{i+1,j+1}}) \\
&= \lambda^{F(B)} \mu^{G(B)} D(x[1])^B s(\lambda)^{-F(A_{\min})} t(\mu)^{-G(A_{\max}^*)} s(x[1])^{-A_{\min}} \prod_{B_{i,j}=-1} \frac{D(x[1]_{i,j})}{x[1]_{ij} s(x[1]_{i,j})} \\
&= \lambda^{F(B)} \mu^{G(B)} s(\lambda)^{-F(A_{\min})} t(\mu)^{-G(A_{\max}^*)} s(x[1])^{-A_{\min}} \prod_{B_{i,j}=1} D(x[1]_{i,j}) \prod_{B_{i,j}=-1} \frac{1}{x[1]_{ij} s(x[1]_{i,j})} \\
& \quad (11)
\end{aligned}$$

Comparing equation (10) with equation (11), we must show that:

$$\begin{aligned}
& s(\lambda)^{F(A_{\min})} \lambda^{F(C_{\max})} t(\mu)^{G(A_{\max}^*)} \mu^{G(C_{\min}^*)} x[1]^{C_{\max}} s(x[1])^{A_{\min}} \\
&= s(\lambda)^{F(B)} \lambda^{F(B)} t(\mu)^{G(B)} \mu^{G(B)} (x[1] s(x[1]))^B
\end{aligned}$$

To complete the proof one need only observe that:

$$\min(x+1, y) + \max(x, y-1) = x + y$$

More precisely, we have, by construction, that:

$$\begin{aligned}
\bar{A}_{i,j}^{\min} &= \max(\bar{B}_{i,j}, \bar{B}_{i+1,j+1} - 1) \\
\bar{C}_{i,j}^{\max} &= \min(\bar{B}_{i-1,j-1} + 1, \bar{B}_{i,j})
\end{aligned}$$

and so:

$$\begin{aligned}
\bar{C}_{i,j}^{\max} + \bar{A}_{i-1,j-1}^{\min} &= \min(\bar{B}_{i-1,j-1} + 1, \bar{B}_{i,j}) + \max(\bar{B}_{i-1,j-1}, \bar{B}_{i,j} - 1) \\
&= \bar{B}_{i,j} + \bar{B}_{i-1,j-1}
\end{aligned}$$

This gives us the same power of $\lambda_{i,j}$ on both sides. By equations 2 and 3 we also have:

$$C_{i,j}^{\max} + A_{i-1,j-1}^{\min} = B_{i,j} + B_{i-1,j-1}$$

This gives us the same power of $x_{i,j}$ on both sides.

The power of $\mu_{i,j}$ on the left hand side is given by:

$$\begin{aligned}
\underline{C}_{i,k-j+1}^{*\min} + \underline{A}_{i-1,k-j}^{*\max} &= \max(\underline{B}_{i,k-j+1}, \underline{B}_{i+1,k-j+2}) + \min(\underline{B}_{i,k-j+1}, \underline{B}_{i+1,k-j+2}) \\
&= \underline{B}_{i,k-j+1} + \underline{B}_{i+1,k-j+2}
\end{aligned}$$

which is the power of $\mu_{i,j}$ on the right hand side. The result follows. \square

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